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Some Aspects of Electromagnetic Waves Propagating in Nonlinear Media taking Gravity into account

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Abstract: In this paper we propose to solve approximately using perturbation theory, Maxwell's equations in general relativity ie in curved space-time (that describes gravitational effects of matter on the electromagnetic field) taking in addition account of the fact that the medium may be nonlinear, inhomogeneous and anisotropic that is described by a electromagnetic field dependent permittivity – permeability –conductivity tensor. We also propose a method for estimating under such circumstances, ie gravitational effects & inhomogeneity anisotropicity & nonlinearity of the medium into account, the surface current density induced on an antenna when an electromagnetic field is incident upon it. The entire formalism is based on the tensor calculus and covariant differentiating fundamental to the general theory of relativity.

1. Introduction

Charge particle dynamics in the presence of strong magnetic fields and gravity is an area of astrophysics. The effect of gravity on magnetic field and electric field nonlinearity is studied [1] with field energy momentum of the particle. The propagation of X rays and Gamma rays was also tested in strong magnetic and gravitational fields that shows birefringence [2].It was also proved by the delay in the modes [2].The propagation of electromagnetic field and gravitational fields in plasma with an emphasis on nonlinear effects and applications in Astrophysics was studied [3].Three dimensional magnetic solutions were also studied in the presence of gravity and nonlinear electromagnetic fields [4].
Here in this paper using tensor calculus parameters of a medium are estimated considering gravitational effects and mathematical technique is given to calculate antenna current density.

3. Mathematical formulae

In general relativity, the permittivity permeability conductivity tensor in the frequency domain is a 4th rank tensor $\varepsilon_{\propto\beta}^{\mu V}(\omega, \underline{r}\,)$.If $A_{\mu}(\omega, \underline{r})$ are the covariant four potentials in the frequency domain, then the antisymmetric electromagnetic field covariant tensor is-

$$F_{\mu v}(w, \underline{r}) = A_{v,\mu} - A_{\mu,v}$$
$$\text{Where} \quad A_{v,o} \equiv j\omega A_v\,(\omega,\underline{r})$$

The gravitational field is assumed to be state i.e. this metric coefficients are time independent.

Thus the electric field components are given by

$$F_{ro} = A_{o,r}\text{-}\, j\omega\, A_r = -E_r(\omega,\underline{r}), \qquad r = (1, 2, 3)$$

and the magnetic fields components are

$$F_{12} = A_{2,1}\ \text{-}\ A_{1,2} = -B_3\,(\omega, \underline{r}\,)$$
$$F_{23} = A_{3,2}\ \text{-}\ A_{2,3} = -B_1(\omega, \underline{r}\,)$$
$$F_{31} = A_{1,3}\ \text{-}\ A_{3,1} = -B_2\,(\omega, \underline{r}\,)$$

The contravariant field tensor

$$\tilde{F}^{\mu v}(\omega, \underline{r}) = \in_{\propto\beta}^{v,\mu}\,(\omega, \underline{r})F^{\propto\beta}(\omega, \underline{r}\,) \tag{1}$$

It has components $\vec{D}(\omega, \underline{r}\,)$, $\vec{H}(\omega, \underline{r}\,)$

It further, the medium is nonlinear, its components will be

$$\tilde{F}^{\mu v}(\omega, \underline{r}) = \in_{\propto\beta}^{\mu v}\,(\omega, \underline{r}\,, F\,(\omega, \underline{r})]\quad F^{\propto\beta}(\omega, \underline{r}\,) \tag{2}$$

Where now the permittivity, permeability and conductivity tensor $\epsilon^{\mu v}_{\propto\beta}$ is field dependent i.e. depends on $\underline{F}(\omega,\underline{r})= \gamma F^{\propto\beta}(\omega,\underline{r})$ nonlinearly.

The Maxwell equations

$$\text{div } \underline{D} = \rho \tag{4}$$

$$\text{Curl } \underline{H} = \underline{J} + j\omega\, \underline{D} \tag{5}$$

in the presence of gravity assume the form

$$(\tilde{F}^{\mu v}(\omega,r)\sqrt{-g(\underline{r})})_{,v} = J^{\mu}(\omega,\underline{r})\sqrt{-g(\underline{r})} \tag{6}$$

Or

$$j\omega\, \tilde{F}^{\mu 0}(\omega,r)\sqrt{-g(\underline{r})} + (\tilde{F}^{\mu m}(\omega,r)\sqrt{-g(\underline{r})})_{,m} = J^{\mu}(\omega,\underline{r})\sqrt{-g(\underline{r})} \tag{7}$$

Taking $\mu = 0$ gives the general relativistic version of Ampere's law (2) in a non-linear medium

$$\cup\ \omega\, \tilde{F}^{r0}(\omega,r)\sqrt{-g(\underline{r})} + (\tilde{F}^{rm}(\omega,r)\sqrt{-g(\underline{r})})_{,m} = J^{r}(\omega,\underline{r})\sqrt{-g(\underline{r})} \tag{8}$$

as that of Gauss' law is

$$(\tilde{F}^{0m}(\omega,r)\sqrt{-g(\underline{r})})_{,m} = \rho(\omega,\underline{r})\sqrt{-g(\underline{r})} \tag{9}$$

To solve these equations approximately using Perturbation theory, we expand $\in^{\mu v}_{\rho\sigma}$ $\omega.r.t.$ the EM field $\underline{F}$ upto second order (Kerr nonlinearity)

$$\in^{\mu v}_{\alpha\beta}(\omega,r,F(\omega,\underline{r})) \approx \in^{\mu v(0)}_{\alpha\beta}(\omega,r) + \delta \in^{\mu v\,(1)}_{\alpha\beta\rho\sigma}(\omega,r)F^{\rho\sigma}(\omega,\underline{r}) + \delta^2 \int \in^{\mu v\,(2)}_{\alpha\beta\rho_1\sigma_1\rho_2\sigma_2}(\omega_1, \omega\text{-}\omega_1, \underline{r})F^{\rho_1\sigma_1}(\omega_1,\underline{r})F^{\rho_2\sigma_2}(\omega-\omega_1,\underline{r})d\,\omega_1 \tag{10}$$

Formally, we write this relation as

$$\in^{\mu v}_{\alpha\beta}[(\omega,\underline{r},F(\omega',\underline{r})] = \in^{\mu v(0)}_{\alpha\beta}(\omega,r) + \delta \in^{\mu v\,(1)}_{\alpha\beta}.F(\omega,\underline{r}) + \delta^2 \in^{\mu v\,(1)}_{\alpha\beta}.(F \otimes F)(\omega,\underline{r}) + \mathrm{O}(\delta^3) \tag{11}$$

We now apply perturbation theory to this

$$F^{\mu v}(\omega,\underline{r}).\, F^{\mu v\,(0)}(\omega,\underline{r}) + \delta.F^{\mu v(1)}(\omega,\underline{r}) + \delta^2.F^{\mu v(2)}(\omega,\underline{r}) + \mathrm{O}(\delta^3) \tag{12}$$

Then the O (δ^0) equation is

$$(\in^{\mu v(0)}_{\alpha\beta}(\omega,\underline{r})\, F^{\alpha\beta(0)}(\omega,\underline{r}).\sqrt{-g(\underline{r})})_{,v} = J^{\mu} \tag{13}$$

Then the O (δ^1) equation is

$$(\in^{\mu v(0)}_{\alpha\beta}(\omega,\underline{r})F^{\alpha\beta(1)}(\omega,\underline{r})\sqrt{-g})_{,v} = 0 + (\in^{\mu v(1)}_{\alpha\beta} F^{(0)}(\omega,\underline{r})]\, F^{\alpha\beta(0)}(\omega,\underline{r}).\sqrt{-g}))_{,v} = 0 \tag{14}$$

And the O (δ^2) equation is

$$(\in^{\mu v(0)}_{\alpha\beta}(\omega,\underline{r})F^{\alpha\beta(2)}(\omega,\underline{r})\sqrt{-g})_{,v} + [(\in^{\mu v(1)}_{\alpha\beta}.F^{(1)}(\omega,\underline{r}))\, F^{\alpha\beta(0)}(\omega,\underline{r}).\sqrt{-g})]_{,v} + [(\in^{\mu v(2)}_{\alpha\beta}.F^{(0)}(\omega,\underline{r})]\, F^{\alpha\beta(0)}(\omega,\underline{r}).\sqrt{-g})_{,v} + [(\in^{\mu v(1)}_{\alpha\beta}.F^{(0)}(\omega,\underline{r})]\, F^{\alpha\beta(1)}(\omega,\underline{r}).\sqrt{-g})_{,v} = 0 \tag{15}$$

(13), (14), (15) can be expressed by separately the two parts as

$$\text{jw}\in^{\mu 0(0)}_{\alpha\beta}\left(\omega,\ \underline{r}\right)F^{\alpha\beta(0)}\left(\omega,\underline{r}\right).\ \sqrt{-g(r)}$$
$$+\ (\in^{\mu m(0)}_{\alpha\beta}\left(\omega,\ \underline{r}\right)F^{\alpha\beta}\left(\omega,\underline{r}\right).\ \sqrt{-g(r)}\,)_{,\,m} = J^{\mu}(\omega,r) \qquad (16)$$

$$\text{j}\omega(\in^{\mu 0(0)}_{\alpha\beta}\left(\omega,\ \underline{r}\right)F^{\alpha\beta(1)}\left(\omega,\underline{r}\right).\ \sqrt{-g(r)}\ )$$
$$+\ (\in^{\mu m(0)}_{\alpha\beta}\ F^{\alpha\beta(1)}\left(\omega,\underline{r}\right).\sqrt{-g(r)}\ )_{,\,m}$$
$$+\ \text{j}\omega\ [(\in^{\mu 0(1)}_{\alpha\beta}\ F^{(0)}\left(\omega,\underline{r}\right).\ F^{\alpha\beta(0)}\left(\omega,\underline{r}\right).\sqrt{-g})]$$
$$+\ \ [(\in^{\mu m(1)}_{\alpha\beta}\ F^{(0)}\left(\omega,\underline{r}\right)]\ F^{\alpha\beta(0)}\left(\omega,\underline{r}\right).\sqrt{-g})]_{,m} = 0 \qquad (17)$$

$$\text{j}\omega\in^{\mu o(0)}_{\alpha\beta}\left(\omega,\underline{r}\right).\ F^{\alpha\beta(2)}\left(\omega,\underline{r}\right)\ +\ \in^{\mu m(0)}_{\alpha\beta}\left(\omega,\underline{r}\right).\ F^{\alpha\beta(2)}\left(\omega,\underline{r}\right),m$$
$$+\ \text{other terms is (15)} = 0 \qquad (18)$$

Let us restrict ourselves to O (δ). Then We've to solve just (16) & (17).

These are to be combined with the Maxwell equations

$$F_{\mu\nu,\alpha}+F_{\nu\alpha,\mu}+F_{\alpha\mu,\nu}=0, \qquad (19)$$

Or
$$F_{01,2}+F_{12,0}+F_{20,1}=0,$$
$$F_{01,3}+F_{13,0}+F_{30,1}=0, \qquad .(20)$$
$$F_{02,3}+F_{23,0}+F_{30,1}=0,$$
$$F_{12,3}+F_{23,1}+F_{31,2}=0, \qquad (21)$$

Or equivalent
$$F_{\mu\nu}=\ A_{\nu,\mu}-\ A_{\mu,\nu} \qquad (22)$$

We impose c gauge condition

$$(A^{\mu}\sqrt{-g}\,)_{,\mu}=0 \qquad (23)$$

Which is the same as the tensor equation

$$A^{\ \mu}_{\ :\mu}=0$$

Writing
$$A^{\mu}\ =\ A^{\mu\,(0)}+\ \delta\,A^{\mu\,(1)}+O\,(\delta^2), \qquad (24)$$

This become

$$(A^{\mu(0)}\sqrt{-g}\,)_{,\mu}=0\,, \qquad (25)$$
$$(A^{\mu(1)}\sqrt{-g}\,)_{,\mu}=0\,, \qquad (26)$$

Now,
$$A^{\mu}\ =\ g^{\mu\nu}\ \ A_{\nu}$$

So,
$$A^{\mu\,(0)}\ =\ g^{\mu\nu}\ A^{\ (0)}_{\ \nu},$$
$$A^{\mu\,(1)}\ =\ g^{\mu\nu}\ A^{\ (1)}_{\ \nu},$$
$$F^{\ (0)}_{\ \mu\nu}=\ A^{\ (0)}_{\ \nu,\mu}\ -\ A^{\ (0)}_{\ \mu,\nu}$$
$$F^{\ (1)}_{\ \mu\nu}=\ A^{\ (1)}_{\ \nu,\mu}\ -\ A^{\ (1)}_{\ \mu,\nu}$$

Then,
$$F^{\alpha\beta(0)}\ =\ g^{\alpha\mu}\ \ g^{\beta\nu}F^{\ (0)}_{\ \mu\nu}$$
$$F^{\alpha\beta(1)}\ =\ g^{\alpha\mu}\ \ g^{\beta\nu}F^{\ (1)}_{\ \mu\nu}$$

We are assuming the gravitational field to be strong so we do not make any per perturbation experiment of $g_{\mu\nu}$. To solve (16) & 17, we must solve a "Green's Function equation" of the kind

$$((\in^{\mu\nu(0)}_{\alpha\beta}\left(\omega,\underline{r}\right)\sqrt{-g(r)}\ \ F^{\alpha\beta(\omega,\underline{r})})_{,\nu}\ =J^{\mu}\left(\omega,\underline{r}\right) \qquad (27)$$

Or equivalently

$$(\in^{\mu\nu(0)}_{\alpha\beta}\left(\omega,\underline{r}\right)\sqrt{-g\ (\underline{r})}\ \ g^{\alpha\rho}\left(\underline{r}\right)\ \ g^{\beta\sigma}\left(\underline{r}\right)F_{\rho\sigma}\left(\omega,\underline{r}\right))_{,\,\nu}=J^{\mu\ (\omega,\underline{r})} \qquad (28)$$

Or $\left(X^{\mu\nu\rho\sigma}\left(\omega,\underline{r}\right),F_{\rho\sigma}\left(\omega,\underline{r}\right)\right)_{,\,\nu}=J^{\mu}\ \left(\omega,\underline{r}\right)$ (29)

Where $F_{\rho\sigma} = A_{\sigma,\rho} - A_{\rho,\sigma}$

$$X^{\mu\nu\rho\sigma} = \in^{\mu\nu(0)}_{\alpha\beta} \sqrt{-g} \ g^{\alpha\rho} \ g^{\beta\sigma} \tag{30}$$

This is equivalents to solving

$$(X^{\mu\nu\rho\sigma}(A_{\sigma,\rho} - A_{\rho,\sigma}), \nu = J^{\mu} \tag{31}$$

If the gravitational field is not very strong, we can expand

$$g_{\mu\nu} = \eta_{\mu\nu} + \delta g_{\mu\nu} \ (r) \tag{32}$$

Where $(\eta_{\mu\nu}) = diag\ [1, -1, -1, \ -1]$ is the flat space time Minkowki metric and then

$$X^{\mu\nu\rho\sigma} = \eta^{\mu\rho}\eta^{\nu\sigma} + \delta X^{\mu\nu\rho\sigma}(\omega, \underline{r}) \tag{33}$$

Where we assume that $\in^{\mu\nu(0)}_{\rho\sigma} \approx \delta^{\mu}_{\rho}\delta^{\mu}_{\sigma} + \delta \in^{\mu\nu(0)}_{\rho\sigma}$ (34)

i.e. the medium is only weakly inhomogeneous and weakly anisotropic.

This gives with $F_{\rho\sigma} = F^{(0)}_{\rho\sigma} + \delta F_{\rho,\sigma}$, $F^{\mu\nu(0)}_{,\nu} = J^{\mu}$, (35)

$$F^{\mu\nu\,(0)} = \eta^{\mu\rho}\eta^{\nu\sigma}F^{(0)}_{\rho\sigma}$$

$$= A^{(0)\nu,\mu} - A^{(0)\,\mu\,,\nu} \tag{36}$$

Where raising and lowering of the Indices γ are taken w.r.t. the Minkowski Metric.

We get the Zero[th] and first order equations from

$$[(\eta^{\mu\rho}\eta^{\nu\sigma} + \delta\chi^{\mu\nu\rho\sigma}(\omega, \underline{r}) \ (F^{(0)}_{\rho\sigma} + \delta F^{\rho\sigma})]_{,\nu} = J^{\mu} \tag{37}$$

i.e. $F^{(0)\mu\nu}_{1\nu} = J^{\mu}$,

$$\delta F^{(0)\mu\nu}_{,\nu} = -\left(F^{(0)}_{\rho\sigma}\ \delta X^{\mu\nu\rho\sigma}\right)_{,\nu} \tag{38}$$

The Gauge condition is $(A^{\mu} \sqrt{-g)}\,)_{,\nu} = 0$

Or equivalently up to first order of smallness $((A^{(0)\mu} + \delta A^{\mu})(1 + \frac{1}{2}\delta g))_{,\mu} = 0$ (39)

$$(\delta g = \delta g^{\propto}_{\propto} = \eta_{\propto\beta} \ \delta g_{\propto\beta}) \tag{40}$$

$$\text{Or } A^{(0)\mu}_{,\mu} = 0, \qquad \delta A^{\mu}_{,\mu} - \frac{1}{2}(A^{(0)\mu}\ \delta g)_{,\mu} \tag{41}$$

Using this we solve

$$F^{(0)\mu\nu}_{,\nu} = (A^{(0)\nu,\mu} - A^{(0)\mu,\nu})_{,\nu} \tag{42}$$

$$= -A^{(0)\mu,\nu}_{,\nu} = J^{\mu}$$

$$\text{i e } \Box A^{(0)\mu} = -J^{\mu} \tag{43}$$

$$A^{(0)\mu}(t, r) = \frac{1}{4\pi}\int \frac{J^{\mu}(t - \mid \underline{r} - \underline{r'} \mid, r')}{- \mid \underline{r} - \underline{r^1} \mid} \ d^3r' \tag{44}$$

$$\text{Or } A^{(0)}(X) = -\int G(X - X^1)J^{\mu}(X')d^4 \ X^1 \tag{45}$$

$$\delta A^{\nu,\mu}{}_{,\nu} - \delta A^{\nu,\mu}{}_{,\nu} = -\left(F^{(0)}_{\rho\sigma}\delta\chi^{\mu\nu\rho\sigma}\right)_{,\nu} - \delta A^{\nu,\mu}{}_{,\nu} - \frac{1}{2}(A^{(0)\alpha}\ \delta g)^{,\mu}_{,\propto} \tag{46}$$

$$= -\left(F^{(0)}_{\rho\sigma}\delta\chi^{\mu\nu\rho\sigma}\right)_{,\nu} \tag{47}$$

$$\Box\delta A^{\mu} = (F^{(0)}_{\rho\sigma}\delta\lambda^{\mu\nu\rho\sigma})_{,\nu} - \frac{1}{2}(A^{(0)\propto}\delta g)^{,\mu}_{,\alpha} \tag{48}$$

So that $\delta A^{\mu}(t, r) = \delta A^{\mu}(x)$

$$=\int G(X-X')\,[(F^{(0)}_{\rho\sigma}(X')\delta\,\chi^{\mu\nu\sigma\rho}(X',1)_{,\nu}\ -\frac{1}{2}(A^{(0)\propto}(X')\delta g(X')^{,\mu}_{,\alpha}]d^4X' \qquad (49)$$

where $G(x)=\mathcal{F}_4^{-1}(-\frac{1}{k_\propto h^\propto}\triangleq\mathcal{F}_4^{-1}(\frac{-1}{k^2})$ (50)

$$=\frac{\delta(X_\propto X^\propto)}{2\pi}=\frac{-\delta(X^2)}{2\pi} \qquad (51)$$

$$=\frac{-\delta(t^2-r^2)}{2\pi r}=\frac{-\delta(t-r)}{4\pi r} \qquad (52)$$

Once $A^{\mu}+\delta A^{\mu}$ has been computed in terms of the antenna surface current J_s^r, we can compute the electric field corresponding to this 4 potential in terms of J_s^r,add it to the incident electric field on the antenna and equate its tangential component to zero on the antenna surface.In this way, we obtain an integral equation for the antenna surface current density.

7. References

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