A simple approach to invisibility under different linear approximations

Roberta Palmeri*, and Tommaso Isernia
DIIES, Università Mediterranea di Reggio Calabria
via Graziella, Loc. Feo di Vito, 89124, Reggio Calabria, Italy

Abstract

In this contribution, we exploit inverse scattering and the spectral properties pertaining to different linear approximated solution approaches to design invisibility cloaks. By paralleling results holding for the well-known Born approximation (BA), a Fourier based analysis is derived for the Contrast Source Extended Born and the Strong Permittivity Fluctuation approximations. Then, the derived arguments are conveniently exploited to reach invisibility and a first numerical assessment is performed in case of BA by exploiting the "alternate projections" method.

1. Introduction

The concept of invisibility always had great interest. The current idea of making objects invisible is pursued by several research groups around the world, since it has been observed that is possible to "hide" a given object, making it effectively invisible to the electromagnetic radiation, by means of ad-hoc materials. In this respect, the new concept of invisibility cloaking has been recently proposed.

Several invisibility techniques have been developed in the last years by looking at different kinds of objects. Among them, the Scattering Cancellation (SC) [1] and the Transformation Optics (TO) [2] theories are worth to be mentioned.

The SC is based on the cancellation effect of the polarization vector induced in the cloaking system by covering the object with a volumetric material (cloak). Notably, such a goal is accomplished in case of small (with respect to the wavelength) scatterer, since the cancellation of the first order harmonics of the scattered field is pursued [1]. As a consequence, the cloaking is object-dependent and it is valid under a quasi-static limit approximation.

Conversely, the TO is able to exclude fields from the interior of an invisible system while bending the radiation just outside. Interestingly, the invisibility device is object independent, as any object located in the excluded region of the space will be invisible. Unfortunately, the price to pay is that the constitutive parameters of coats should change continuously point by point and require the cloaking material to be inhomogeneous and strongly anisotropic [3].

In this contribution, a simple approach to reach invisibility is proposed within the inverse scattering framework. The solution of an inverse scattering problem (ISP) is not a trivial task due to the intrinsic non linearity and ill-posedness of the problem itself [4], so that regularization techniques are needed to restore well-posedness. To counteract the non-linearity, the adoption of linearizing approaches seems instead to be attractive. Among them, the well-known Born approximation (BA) [5] which is valid for weak scatterers, is the common one and it has given rise to a number of different approximations by looking at different classes of objects.

By considering the spectral properties of the unknown function involved in the approximated approach at hand, in the following we will show how they can be conveniently exploited for invisibility devices design.

In particular, the electromagnetic model pertaining to the Contrast Source Extended Born (CS-EB) [6,7] and the Strong Permittivity Fluctuation (SPF) [8] are firstly recalled; then, a Fourier based analysis will be derived for the arising linear approximations CS-EBA and SPFA. Finally, such an analytical study will be used to design an invisibility cloak by means of a very simple procedure.

2. The CS-EB model and approximation

As is well known, the ISP within the BA is viable and accurate when weak scatterers are looking for [5]. In this case, the unknown total field inside the investigation domain Ω is approximated by the incident field E_i and hence the problem is linearized. An improvement over the BA is achieved in the (contrast source) extended Born model [6], for which the pertaining Born series is derived in the following.

Let consider the *state* equation governing the ISP, for the 2D TM scalar problem (the factor exp $(j\omega t)$ is assumed and dropped):

$$W(\boldsymbol{r},\boldsymbol{k_t}) = \chi(\boldsymbol{r})E_i(\boldsymbol{r},\boldsymbol{k_t}) + \chi(\boldsymbol{r})\int_\Omega g(\boldsymbol{r},\boldsymbol{r}')W(\boldsymbol{r}',\boldsymbol{k_t})d\boldsymbol{r}' \tag{1}$$

wherein $\boldsymbol{k_t}$ indicates the vector of the incidence direction, $E_i(\boldsymbol{r},\boldsymbol{k_t}) = e^{-j\boldsymbol{k_t}\cdot\boldsymbol{r}}$ is the incident field, χ is the unknown contrast function, $W = E\chi$ is the auxiliary unknown contrast source, E being the total field, while $g(\boldsymbol{r},\boldsymbol{r}') = -j/4H_0^{(2)}(k_b|\boldsymbol{r}-\boldsymbol{r}'|)$ is the Green's function of the free-space, k_b being the pertaining wavenumber and $\boldsymbol{r} = (x,y) \in \Omega$.

By adding and subtracting $W(\boldsymbol{r},\boldsymbol{k_t})$ into the integral operator, eq.(1) recast as:

$$W(\boldsymbol{r},\boldsymbol{k_t}) = \chi(\boldsymbol{r})E_i(\boldsymbol{r},\boldsymbol{k_t}) + \chi(\boldsymbol{r})W(\boldsymbol{r},\boldsymbol{k_t})f_\Omega(r) + \chi(\boldsymbol{r})A_{iMOD}[W(\boldsymbol{r},\boldsymbol{k_t})] \tag{2}$$

where $f_\Omega(\boldsymbol{r}) = \int_\Omega g(\boldsymbol{r}-\boldsymbol{r}')\,dr'$, $A_{iMOD}[W(\boldsymbol{r},\boldsymbol{k_t})] = \int_\Omega g(\boldsymbol{r}-\boldsymbol{r}')\,[W(\boldsymbol{r}',\boldsymbol{k_t}) - W(\boldsymbol{r},\boldsymbol{k_t})]d\boldsymbol{r}'$.

By formally inverting eq.(2) and defining:

$$p(\boldsymbol{r}) = \frac{\chi(\boldsymbol{r})}{I - \chi(\boldsymbol{r})f_\Omega(\boldsymbol{r})} \tag{3}$$

one achieves the following relation, I being the identity operator:

$$W(\boldsymbol{r},\boldsymbol{k_t}) = (I - p(\boldsymbol{r})A_{iMOD}[W(\boldsymbol{r},\boldsymbol{k_t})])^{-1}p(\boldsymbol{r})E_i(\boldsymbol{r},\boldsymbol{k_t}) \tag{4}$$

If $\|p(\boldsymbol{r})A_{iMOD}[W(\boldsymbol{r},\boldsymbol{k_t})]\| < 1$, $\|\cdot\|$ being the ℓ_2-norm, a series expansion for the inverse operator can be performed, thus obtaining the CS-EB series:

$$W(\boldsymbol{r},\boldsymbol{k_t}) = \sum_{n=0}^{+\infty}(p(\boldsymbol{r})A_{iMOD}[W(\boldsymbol{r},\boldsymbol{k_t})])^n\,p(\boldsymbol{r})E_i(\boldsymbol{r},\boldsymbol{k_t}) \tag{5}$$

From the singular nature of the Green's function when $\boldsymbol{r}' = \boldsymbol{r}$, one may expect that the last term in eq.(2) can be neglected; by referring to the CS-EB series (5), it means that the first term is the dominant one and the CS-EB approximation (CS-EBA) is derived as:

$$W(\boldsymbol{r},\boldsymbol{k_t}) = p(\boldsymbol{r})E_i(\boldsymbol{r},\boldsymbol{k_t}) \tag{6}$$

Finally, by substituting eq.(6) into the *data* equation of the ISP, one achieves:

$$E_s(\boldsymbol{k_o},\boldsymbol{k_t}) = A_e[W(\boldsymbol{r},\boldsymbol{k_t})] = A_e[p(\boldsymbol{r})E_i(\boldsymbol{r},\boldsymbol{k_t})] \tag{7}$$

wherein E_s is the scattered field, $\boldsymbol{k_o}$ is the vector of the observation direction and A_e is a short notation for the integral radiation operator.

As it can be easily guessed from eq.(7), the ISP is now linear (but still ill-posed) with respect to the auxiliary function $p(\boldsymbol{r})$ [7].

3. The SPF model and approximation

The SPF approximation (SPFA) has been introduced by Tsang and Kong [8] by exploiting the singularity of the Green's function for the solution of the vectorial problem of wave scattering by random medium, in case of both small and large variance of the permittivity function.

Let us consider the *state* equation (for the electric field) for the 2D vectorial case:

$$\boldsymbol{E}(\boldsymbol{r},\boldsymbol{k_t}) = \boldsymbol{E_i}(\boldsymbol{r},\boldsymbol{k_t}) + \int_\Omega \bar{\bar{\boldsymbol{G}}}(\boldsymbol{r},\boldsymbol{r}')\Delta k^2 \cdot \boldsymbol{E}(\boldsymbol{r}',\boldsymbol{k_t})d\boldsymbol{r}' \tag{8}$$

in which $\bar{\bar{\boldsymbol{G}}}(\boldsymbol{r},\boldsymbol{r}') = \left(\bar{\bar{\boldsymbol{I}}} - \frac{1}{k_b^2}\nabla\nabla\right)g(\boldsymbol{r},\boldsymbol{r}')$ is the dyadic Green's function, $\bar{\bar{\boldsymbol{I}}}$ is the 2D identity operator, $\Delta k^2 = k^2 - k_b^2$ with k the wavenumber of the random medium, while the symbol $(\cdot)$ states for the inner product.

As it is known, $g(\boldsymbol{r},\boldsymbol{r}')$ becomes singular when the observation point is inside the source region and therefore the integral expression (8) gives rise to difficulties. In order to overcome them, by following the approach in [9], the singularity can be treated by means of the *principal volume method*, so that one achieves:

$$\left(\bar{\bar{\boldsymbol{I}}} + \frac{\Delta k^2}{k_b^2}\bar{\bar{\boldsymbol{L}}}\right)\cdot \boldsymbol{E}(\boldsymbol{r},\boldsymbol{k_t}) = \boldsymbol{E_i}(\boldsymbol{r},\boldsymbol{k_t}) + P.V.\int_\Omega \bar{\bar{\boldsymbol{G}}}(\boldsymbol{r},\boldsymbol{r}')\Delta k^2 \cdot \boldsymbol{E}(\boldsymbol{r}',\boldsymbol{k_t})d\boldsymbol{r}' \tag{9}$$

where $P.V.\int_\Omega$ stands for a shape dependent principal value integral and $\bar{\bar{\boldsymbol{L}}}$ is a dyad depending on the shape of the considered volume.

In a more compact form, the state equation for the SPF model reads:

$$\begin{aligned}\boldsymbol{F}(\boldsymbol{r},\boldsymbol{k_t}) &= \boldsymbol{E_i}(\boldsymbol{r},\boldsymbol{k_t}) + P.V.\int_\Omega \bar{\bar{\boldsymbol{G}}}(\boldsymbol{r},\boldsymbol{r}')\bar{\bar{\boldsymbol{I}}}q(\boldsymbol{r}')\cdot \boldsymbol{F}(\boldsymbol{r}',\boldsymbol{k_t})d\boldsymbol{r}' \\ &= \boldsymbol{E_i}(\boldsymbol{r},\boldsymbol{k_t}) + A_{iSPF}\left[\bar{\bar{\boldsymbol{I}}}q(\boldsymbol{r}')\cdot \boldsymbol{F}(\boldsymbol{r}',\boldsymbol{k_t})\right]\end{aligned} \tag{10}$$

wherein $\bar{\bar{\boldsymbol{I}}}q(\boldsymbol{r}) = \bar{\bar{\boldsymbol{q}}}(\boldsymbol{r}) = \Delta k^2\left(\bar{\bar{\boldsymbol{I}}} + \chi\bar{\bar{\boldsymbol{L}}}\right)^{-1}$, $\boldsymbol{F}(\boldsymbol{r},\boldsymbol{k_t}) = \left(\bar{\bar{\boldsymbol{I}}} + \chi\bar{\bar{\boldsymbol{L}}}\right)\cdot \boldsymbol{E}(\boldsymbol{r},\boldsymbol{k_t})$.

By following the same steps of the previous Section, the SPF series is defined as:

$$\boldsymbol{F}(\boldsymbol{r},\boldsymbol{k_t}) = \sum_{n=0}^{+\infty}\left(A_{iSPF}\left[\bar{\bar{\boldsymbol{I}}}q(\boldsymbol{r})\right]\right)^n \cdot \boldsymbol{E_i}(\boldsymbol{r},\boldsymbol{k_t}) \tag{11}$$

which is valid for $\left\|A_{iSPF}\left[\bar{\bar{\boldsymbol{I}}}q(\boldsymbol{r})\right]\right\| < 1$.

The SPFA is derived by considering the first term in (11) as the dominant one, so that:

$$\boldsymbol{F}(\boldsymbol{r},\boldsymbol{k_t}) = \boldsymbol{E_i}(\boldsymbol{r},\boldsymbol{k_t}) \tag{12}$$

and by substituting expression (12) into the data equation:

$$\begin{aligned}\boldsymbol{E_s}(\boldsymbol{k_o},\boldsymbol{k_t}) &= A_e\left[\bar{\bar{\boldsymbol{I}}}q(\boldsymbol{r})\cdot \boldsymbol{F}(\boldsymbol{r},\boldsymbol{k_t})\right] \\ &= A_e\left[\bar{\bar{\boldsymbol{I}}}q(\boldsymbol{r})\cdot \boldsymbol{E_i}(\boldsymbol{r},\boldsymbol{k_t})\right]\end{aligned} \tag{13}$$

it becomes linear in the auxiliary unknown function $q(\boldsymbol{r})$.

4. Spectral analysis for CS-EBA and SPFA

For the sake of brevity, a unique analysis is derived for the CS-EBA and SPFA, and for this latter one component of the electric field is considered for simplicity.

Let us consider to deal with far field observations. In this case, the data equation in the integral form reads:

$$E_s(\boldsymbol{k_o},\boldsymbol{k_t}) = C\int_\Omega \eta(\boldsymbol{r}')e^{-j(\boldsymbol{k_t}-\boldsymbol{k_o})\cdot\boldsymbol{r}'}d\boldsymbol{r}' \tag{14}$$

in which C is a constant value arising from the asymptotic expansion for the radiation operator and $\eta(\boldsymbol{r})$ is a generic unknown function (i.e., $p(\boldsymbol{r})$ or $q(\boldsymbol{r})$ in the two cases above).
Let us also evaluate the Fourier transform of $\eta(\boldsymbol{r})$:

$$\mathcal{F}[\eta(\boldsymbol{r})] = \tilde{\eta}(\boldsymbol{K}) = C\int_{\Omega} \eta(\boldsymbol{r}')e^{-j\boldsymbol{K}\cdot\boldsymbol{r}'}d\boldsymbol{r}' \qquad (15).$$

By comparing eqs.(14)-(15) and by referring to a number of experiments $\boldsymbol{k_t}$, one immediately gets that $E_s(\boldsymbol{k_o},\boldsymbol{k_t}) = \tilde{\eta}(\boldsymbol{k_t}-\boldsymbol{k_o})$, namely the scattered field can be considered the restriction of the Fourier transform of $\eta(\boldsymbol{r})$ to the circle with radius $2|\boldsymbol{k_o}| = 2k_b$, the so-called Ewald sphere. A pictorial representation of such a spectral coverage is shown in fig.1(a).
It is worth to note that such a result is an extension of the yet developed arguments for the BA [5] to generic linear approximations. Therefore, such a spectral analysis could be useful in different scenarios by firstly looking at the unknown $\eta(\boldsymbol{r})$ and then going back to the actual unknown $\chi(\boldsymbol{r})$.
Notably, the spectral coverage clearly indicates which kind of profiles can be safely reconstructed, and hence which of them are instead expected to be "invisible". As a matter of fact, it has been derived that the scattered field is related to the spatial Fourier transform of the auxiliary unknown function involved in the approximation at hand, over the surface of a single Ewald sphere. Interestingly, such a circumstance can be conveniently used in case of invisibility problems, namely when the goal of cancelling out the scattered field from an object is pursued. As a consequence, the graphical representation of fig.1(a) can be renewed as in fig.1(b): if the spectral content of the actual unknown is null inside the Ewald sphere, the arising scattering field will be null as well.
As it can be noted, it is possible (at least in principle) to synthesize a cover by looking at scatterers whose spectral content is outside the Ewald sphere. Obviously, care must be taken with the range of validity of the considered approximations.

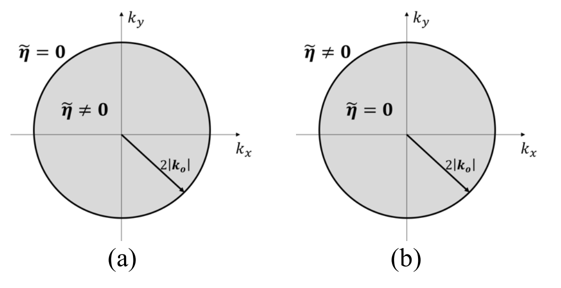

Figure 1. Spectral coverages for linear approximations. (a) "Visibility" and (b) "invisibility" condition.

4. Preliminary assessment

By taking advantage from the simple arguments proposed in the previous Section, a first preliminary assessment is performed by pursuing the invisibility of an object in the BA. Hence, $\eta(\boldsymbol{r}) = \chi(\boldsymbol{r})$ and $\tilde{\eta}(\boldsymbol{K}) = \tilde{\chi}(\boldsymbol{K})$.
Let χ_0 be the contrast function of the lossless object to be hidden (with support Σ_0) and $\Delta\chi$ the contrast function pertaining to the unknown cover (with support Σ), so that the overall scattering system is characterized by $\chi = \chi_0 + \Delta\chi$, see fig.3(a). A simple approach to reach invisibility deals with the exploitation of the "alternate projections" method [10] relying on the iterative projection of the unknown function $\chi \in \mathbb{R}^2$ into the Fourier domain in which all the components belonging to the circle of radius $2k_b$ are forced to be zero, see fig.2.

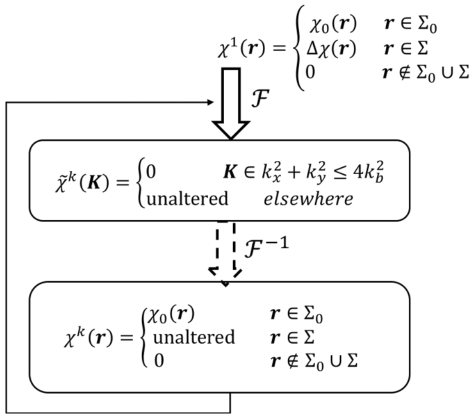

Figure 2. Schematic representation of the "alternate projections" synthesis approach.

The considered object exhibits a varying contrast function from 0.3 to 0 and radius 0.12λ, λ being the wavelength in the free space, see fig.3(a). The investigation domain Ω is a square with side 1.5λ, while the radius of the circular region Σ is equal to 0.6λ. For the starting point of the iterative procedure $\Delta\chi = 0.1$.
The outcome of the alternate projections after 20 iterations is shown in fig.3(b). As it can be seen, starting from the initial case in fig.3(c), the alternate projections iteratively get out the dominant Fourier harmonics of the circle of radius $2k_b$, see fig.3(d). In addition, the field scattered from the so-obtained cloaking system is reduced with respect to the bare case, see figs.3(e)-(f).

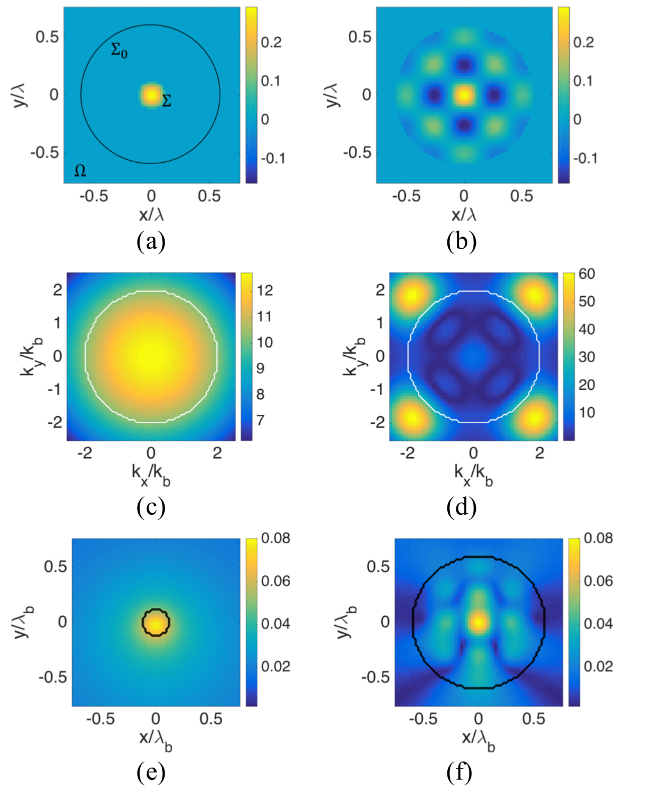

Figure 3. Numerical example for spectral invisibility. Contrast function of the (a) bare and (b) covered object. Fourier transform of the (c) bare ($\tilde{\chi}_0$) and (d) covered ($\tilde{\chi}$) object; the white contour line represents the Ewald sphere with radius $2k_b$. Scattered field from the (e) bare and (f) covered system with superimposed the contour line of the object.

6. Conclusion

In this contribution, the synthesis of invisibility cloaks by exploiting a Fourier based analysis in the inverse scattering framework has been proposed. In particular, a spectral analysis has been derived for linear approximated solution approaches, such as the Contrast Source Extended Born and the Strong Permittivity Fluctuation approximations. Then, the arising relationships between the scattered field and the unknown function encoding the electromagnetic properties of the object have been conveniently exploited for the design of invisible systems. The proposed synthesis method has been assessed within the Born approximation; however, it is quite simple and general, so that it can be conveniently extended to a number of linear models.

Further numerical examples concerning other approximated strategies, as well as different kinds of configuration (see [11,12]) will be shown at the Conference.

7. References

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