URSI GASS 2023, Sapporo, Japan, 19 – 26 August 2023

Efficient Uncertainty Quantification of Deterministic Wireless Channel Models Using Polynomial Chaos Expansion

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Abstract

Uncertainty in the input specification for deterministic channel models such as ray-tracing introduces variations in the predicted signal strength. This necessitates the quantification and analysis of the impact of input uncertainties on the channel models, as a means of ensuring performance robustness. The polynomial chaos expansion (PCE) method has emerged as a promising uncertainty quantification technique compared to the commonly used yet computationally inefficient Monte Carlo methods. However, PCE-based methods generally suffer from a "curse of dimensionality", where the computational cost increases rapidly with the number of random variables included in the analysis. This paper applies an orthogonal matching pursuit algorithm to mitigate the computational cost of PCE and facilitate the uncertainty analysis of ray-tracing based channel models. The performance is demonstrated in an indoor environment and validated against Monte Carlo simulations and experimental measurements.

1 Introduction

Characterization of radio wave propagation is of great importance for the deployment of wireless communication systems in indoor environments [1]. Consequently, there has been significant interest in the development of deterministic models such as ray-tracing to study the propagation characteristics of radio channels. However, the accuracy of such models depends heavily on the description of the modeling environment, which often involves considerable uncertainty. Therefore, it is necessary to quantify the impact of input uncertainties on the output of interest and give a measure of confidence in the predicted results.

The widely used uncertainty quantification techniques are the Monte Carlo (MC) and polynomial chaos expansion (PCE) methods [2]. However, limited by the slow rate of convergence, the MC method is seldom applied to computationally large problems such as indoor propagation. Recently, techniques based on PCE have been coupled with computational electromagnetic methods for the uncertainty analysis of deterministic channel models [3]. The PCE method approximates the output of interest with an expansion of orthogonal polynomial basis functions, where each function is weighted by an expansion coefficient. For a small number of random variables, the associated expansion coefficients can be evaluated with a few deterministic simulations. However, as the number of random variables increases, the computational requirement grows rapidly and the efficiency of PCE is significantly compromised.

This paper applies an orthogonal matching pursuit algorithm [4] for the uncertainty analysis of ray-tracing based channel models. The approach can lead to considerable reduction in the computational requirement by including only a small fraction of polynomials in the PCE approximation. The associated expansion coefficients are evaluated while identifying the necessary polynomials. The effectiveness of the approach is demonstrated in the example of an underground parking garage, where uncertainties in the garage dimensions, the wall materials as well as the spatial locations of transmitting and receiving antennas are considered. A clear improvement in computational efficiency, compared to traditional PCE and MC methods, is achieved.

2 Polynomial Chaos Expansion

2.1 Formulation

The PCE method uses a truncated expansion of orthogonal polynomial basis functions to approximate a random output function of interest, $X(\boldsymbol{\xi})$, as:

$$X(\boldsymbol{\xi}) = \sum_{k \in \mathscr{K}_p} a_k \Psi_k(\boldsymbol{\xi}) \tag{1}$$

where $\boldsymbol{\xi}$ is the vector of random input variables; $\Psi_k(\boldsymbol{\xi})$ is the k-th polynomial basis function and a_k is the associated expansion coefficient; $\mathscr{K}_p$ denotes the set of polynomial basis functions. The mean and variance of the output, $X(\boldsymbol{\xi})$, can be estimated directly through the expansion coefficients and polynomial basis functions [5].

In general, (1) can be applied to characterize the uncertainty of deterministic channel models by expanding either the governing equations or the solutions, which are referred to as *intrusive* and *non-intrusive* approaches, respectively. In this paper, the non-intrusive approach is adopted due to its simple implementation, where the ray-tracing model is used as a "black box".

2.2 Expansion Coefficients

The total number of expansion coefficients in (1) is

$$P = \frac{(M+D)!}{M!D!} \tag{2}$$

where M is the number of random input variables and D is the polynomial order in the expansion [2]. Generally, the expansion coefficient is evaluated through a Galerkin projection approach by utilizing the orthogonality of polynomial basis [5]:

$$a_k = \frac{\langle X(\boldsymbol{\xi}), \Psi_k(\boldsymbol{\xi})\rangle}{\langle \Psi_k^2(\boldsymbol{\xi})\rangle}. \tag{3}$$

Equation (3) can be further evaluated using numerical quadrature rules as:

$$a_k \approx \frac{1}{\langle \Psi_k^2(\boldsymbol{\xi})\rangle} \sum_{q=1}^{Q} X(\boldsymbol{\xi}_q)\Psi_k(\boldsymbol{\xi}_q)w_q \tag{4}$$

where $\boldsymbol{\xi}_q$ is the quadrature point and w_q is the integration weight coefficient. As a result, the evaluation of the expansion coefficients can be cast into a numerical integration problem, where Q deterministic simulations with, respectively, $\boldsymbol{\xi}_1$, $\boldsymbol{\xi}_2$, $\boldsymbol{\xi}_3$, ..., and $\boldsymbol{\xi}_Q$, as inputs are required for the calculation. The commonly used quadrature rules are the Gaussian and Smolyak sparse quadratures [5], where the set of quadrature points and integration weights are uniquely determined by the marginal distributions of the random input variables.

3 Efficient Uncertainty Quantification using Polynomial Chaos with Orthogonal Matching Pursuit

The Smolyak sparse quadrature generally requires substantially fewer deterministic simulations compared to the Gaussian quadrature. However, when M or D is large, the computational cost is still considerable. In general, when the output function of interest is smooth, the PCE exhibits sparsity in that a small fraction of expansion coefficients are significant [6]. Therefore, we could exploit this sparsity and identify the polynomials that are necessary for the PCE approximation. The output function in (1), $X(\boldsymbol{\xi})$, can be approximated as:

$$X(\boldsymbol{\xi}) \approx \sum_{k\in\mathscr{K}_s} a_k\Psi_k(\boldsymbol{\xi}) \tag{5}$$

where $\mathscr{K}_s$ denotes the smallest set of polynomial basis functions that need to be retained and $\mathscr{K}_s \subset \mathscr{K}_p$. As a result, only a small fraction of all the unknown expansion coefficients need to be evaluated, leading to a reduction in the computational requirement.

A natural question is how to discern which polynomial basis functions are appropriate to be included in $\mathscr{K}_s$. This can be solved through the following optimization problem [6]:

$$\min \|\mathbf{a}\|_0 \qquad \text{subject to} \quad \phi\mathbf{a} = \mathbf{y} \tag{6}$$

where $\|\cdot\|_0$ denotes the number of nonzero elements of the argument; $\mathbf{a}$ is a $P\times 1$ vector containing all associated expansion coefficients for the polynomials in $\mathscr{K}_p$; ϕ, referred to as the measurement matrix [6], is an $N\times P$ matrix that maps $\mathbf{a}$ to $\mathbf{y}$; $\mathbf{y}$ is a $N\times 1$ vector containing the sampled output results from N deterministic simulations.

In practice, solutions to the problem (6) can be obtained effectively via the orthogonal matching pursuit (OMP) algorithm [4]. The OMP algorithm is a greedy search approach, which iteratively finds the set of polynomial basis elements that best approximate the sampled output results $\mathbf{y}$. On each iteration, the algorithm includes a new polynomial basis element into an active basis set. The associated expansion coefficients are updated through a least-squares optimization to minimize the approximation residual, $\mathbf{r} = \mathbf{y} - \phi\mathbf{a}$ [4]. The process is repeated until all significant polynomials are included.

When applying such an approach to the uncertainty analysis of deterministic channel models, the whole procedure can be summarized as:

Step 1 Generate N sets of randomly sampled inputs for the deterministic simulations and obtain the corresponding output results $\mathbf{y}$; Set $\mathbf{r} = \mathbf{y}$ and $\mathscr{A} = \varnothing$, where $\mathbf{r}$ is the approximation residual and $\mathscr{A}$ is the index set of the retained polynomials for the PCE approximation.

Step 2 Find the index of the polynomial basis function that is most correlated with the current approximation residual $\mathbf{r}$ [4]:

$$k = \operatorname{argmax}_{j\notin\mathscr{A}} |\langle \mathbf{r}, \phi(:,j)\rangle| \tag{7}$$

where $\phi(:,j)$ denotes all elements at the j-th column of the measurement matrix ϕ.

Step 3 Update the index set as $\mathscr{A} = \mathscr{A} \cup k$ and add the corresponding polynomial basis function, $\Psi_k(\boldsymbol{\xi})$, to the active basis set.

Step 4 Calculate the associated expansion coefficients, $\widehat{\mathbf{a}}$, through a least-squares optimization [4]:

$$\widehat{\mathbf{a}} = \operatorname{argmin} \|\mathbf{y} - \phi\mathbf{a}\|_2 \quad \text{subject to} \quad a_i = 0, \forall i \notin \mathscr{A} \tag{8}$$

Step 5 Update the approximation residual, $\mathbf{r} = \mathbf{y} - \phi\widehat{\mathbf{a}}$, and calculate the relative difference between the sampled output results and the PCE approximation using the following Euclidean error norm:

$$\mathscr{E} = \frac{\sqrt{\frac{1}{N}\|\mathbf{y} - \phi\widehat{\mathbf{a}}\|_2^2}}{\sqrt{\frac{1}{N}\|\mathbf{y}\|_2^2}}. \tag{9}$$

Step 6 Repeat Steps 2–5 until the maximum number of iterations, $I_{\max} = \min(P, N)$, has been reached or $\mathscr{E}$ is above its minimum value for at least 10 % of $I_{\max}$ iterations [7]; Record the minimum error, $\mathscr{E}_{\min}$.

Step 7 Compare $\mathscr{E}_{\min}$ with a preset threshold $\mathscr{E}_{\text{th}}$. If $\mathscr{E}_{\min} \geq \mathscr{E}_{\text{th}}$, increase the number of sampled simulations, $N = N + 1$, and repeat Steps 1–7. Otherwise, terminate.

The minimum N that satisfies the criterion determines the number of required deterministic simulations and the corresponding PCE approximation is used for the uncertainty analysis.

4 Numerical Results

In this section, the approach discussed in the previous section is applied for the uncertainty analysis of ray-tracing based channel models. The considered environment is an empty underground parking garage, as shown in Fig. 1. The measurement is carried out at 2.4 GHz, where the transmitter (Tx) is fixed and the receiver (Rx) is moved along a line as shown in Fig. 1. Vertically polarized, highly directional antennas are used for both the transmitter and receiver.

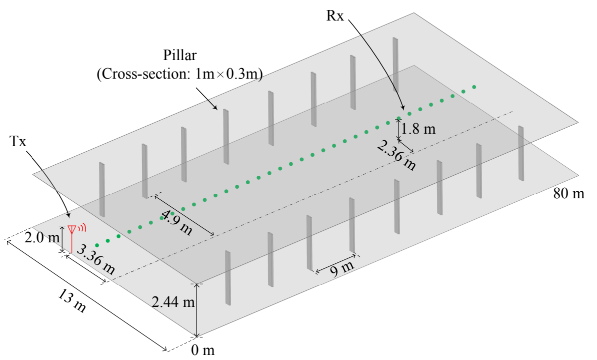

Figure 1. Diagram of the underground parking garage.

An image-based ray-tracer [8] is used to generate the output of interest, which is the received power. Various sources of parameter uncertainty are considered, including randomness in the garage height, the spatial locations of the transmitting and receiving antennas and the wall material properties. All considered uncertain input parameters are summarized in Table 1. The floor and ceiling are formed from poured concrete and thus surface roughness is taken into account.

Following the procedure discussed in the previous section, first the number of RT simulations is determined. Since the output is a vector of the received power at different locations, therefore, the minimum relative error, $\mathscr{E}_{\min}$, is evaluated as the mean of values calculated at all the receiver locations. The received power is sampled at every 0.1 m

Table 1. Uncertain Input Parameters in the RT Model.

Uncertain Input	Nominal Value	Distribution
Garage Height, H	2.44 m	$\sigma = 0.02$
Tx Height, h_{Tx}	2.0 m	$\sigma = 0.03$
Rx Height, h_{Rx}	1.8 m	$\sigma = 0.03$
Tx Lateral Position, l_{Tx}	3.36 m	$\sigma = 0.15$
Rx Lateral Position, l_{Rx}	2.36 m	$\sigma = 0.15$
Relative Permittivity, ε_r	5	$\sigma = 0.5$
Conductivity, σ_0	0.1	$\sigma = 0.03$
Surface Roughness, σ_h	0 m	$\sigma = 0.05$

and a threshold of $\mathscr{E}_{\text{th}} = 1\,\%$ is used throughout the paper. The polynomial order for the PCE approximation at different received locations is chosen adaptively and $D_{\max} = 5$. The trend of the relative error as a function of the number of RT simulations is plotted in Fig. 2. As can be seen, a total number of 31 RT simulations are required.

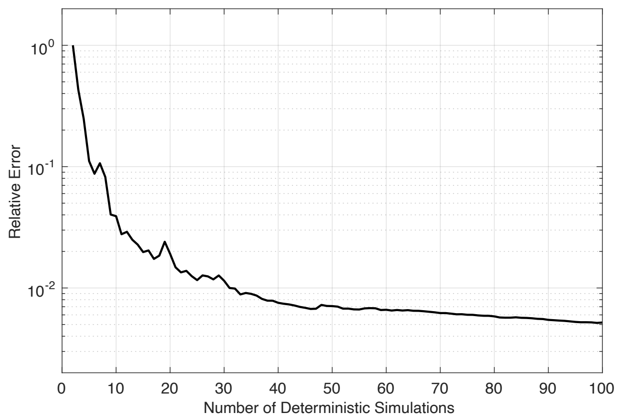

Figure 2. Relative error vs. number of RT simulations.

The mean values and 90 % confidence intervals of the received power computed using our approach are compared against statistics computed using 10000 MC simulations and measured data in Fig. 3. The good agreement demonstrates the validity of the approach. The results in Fig. 3 also show that the variations in the received power can be considerable for relatively small input parameter uncertainties, demonstrating the necessity for the uncertainty analysis of deterministic channel models in indoor environments.

The number of required deterministic simulations and accuracy for different uncertainty quantification techniques are compared in Table 2. In order to numerically quantify the accuracy of different methods, a mean absolute error is evaluated as follows:

$$\mathscr{E}_{\text{MAE}} = \frac{1}{N_0} \sum_{i=1}^{N_0} \left| P(x, y, z_i) - P_{\text{ref}}(x, y, z_i) \right| \tag{10}$$

Table 2. Comparison of the number of deterministic simulations and mean absolute errors for different uncertainty quantification techniques.

Method	Monte Carlo	PCE (Gaussian)			PCE (Smolyak)				PCE (Proposed)
		D = 1	D = 2	D = 3	D = 1	D = 2	D = 3	D = 4	
Number of Deterministic Simulations	10000	256	6561	65536	17	129	609	2193	31
Mean Absolute Error [dB]	—	0.58	0.13	0.11	0.75	0.14	0.12	0.12	0.13

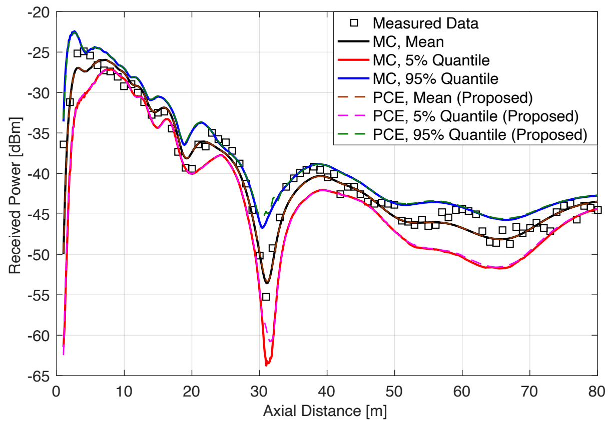

Figure 3. Received power at 2.4 GHz in the underground parking garage.

where $P(x,y,z_i)$ and $P_{\text{ref}}(x,y,z_i)$ is the 90 % confidence intervals of the received power at the (x,y,z_i)-th sampling location obtained from the PCE method and the reference, respectively; $N_0 = 1600$ is the total number of samples; The results generated from 10000 Monte Carlo simulations are used as the reference.

It can be observed that the accuracy of our approach is very close to traditional PCE methods with $D = 2$ or higher. However, the computational requirement is reduced significantly. For example, the number of RT simulations required for our approach is only around 24 % compared to the one using a Smolyak scheme with $D = 2$.

5 Conclusion

The uncertainty quantification of deterministic channel models is generally computationally intensive since a large number of expensive simulation models need be evaluated. This paper presented an efficient approach for the uncertainty analysis of ray-tracing based channel models. The approach utilized the orthogonal matching pursuit algorithm to mitigate the computational cost of PCE methods. Its effectiveness has been demonstrated by comparisons to traditional PCE and MC methods in the example of an underground parking garage, where considerable computational savings were achieved.

References

[1] M. F. Cátedra and J. Perez, *Cell Planning for Wireless Communications*. Artech House, 1999.

[2] O. L. Maitre and O. M. Knio, *Spectral Methods for Uncertainty Quantification: With Applications to Computational Fluid Dynamics*, ser. Scientific Computation. Springer Netherlands, 2012.

[3] A. C. M. Austin, N. Sood, J. Siu, and C. D. Sarris, “Application of polynomial chaos to quantify uncertainty in deterministic channel models,” *IEEE Trans. Antennas Propag.*, vol. 61, no. 11, pp. 5754–5761, November 2013.

[4] J. A. Tropp and A. C. Gilbert, “Signal recovery from random measurements via orthogonal matching pursuit,” *IEEE Trans. Inf. Theory*, vol. 53, no. 12, pp. 4655–4666, December 2007.

[5] D. Xiu, *Numerical Methods for Stochastic Computations: A Spectral Method Approach*. Princeton University Press, 2010.

[6] J. Hampton and A. Doostan, “Compressive sampling of polynomial chaos expansions: Convergence analysis and sampling strategies,” *Journal of Computational Physics*, vol. 280, no. 3, pp. 363–386, 2015.

[7] G. Blatman and B. Sudret, “Adaptive sparse polynomial chaos expansion based on least angle regression,” *Journal of Computational Physics*, vol. 230, no. 6, pp. 2345–2367, 2011.

[8] X. Zhang, N. Sood, J. Siu, and C. D. Sarris, “Calibration of a 3-D ray-tracing model in railway environments,” *IEEE Int. Symp. on Antennas and Propag.*, pp. 89–90, July 2015.